Exam Geometry – 19 June 2019

Note: This exam consists of **four** problems. Usage of the theory and examples of Chapters 1:1-6, 2:1-5, 3:1-3, 4:1-6 of Do Carmo's textbook is allowed. You may not use the results of the exercises, with the exception of the results of Exercise 1-5:2,12, 1-6:2, 4-2:2,3, 4-3:1,2, 4-4:4. Give a precise reference to the theory and/or exercises you use for solving the problems.
You get 10 points for free.

All functions, curves, surfaces, parametrisations and (normal) vector fields in the exam problems are differentiable, i.e., of class C^∞.

Problem 1. (5 + 5 + 5 + 5 = 20 points) Let $\alpha : I \to \mathbb{R}^3$ be a unit-speed curve with constant curvature and zero torsion. We will show that α is part of a circle.

(a) Show that the binormal $\underline{b}$ of α is a constant vector.

(b) Show that α is contained in a plane P.

(c) Show that α lies on a sphere S.
(Hint: use the osculating circle).

(d) Find the centre and the radius of the circle containing α.

Problem 2. (7 + 5 + 5 + 8 = 25 points)
To see the plane $\mathbb{R}^2$ as a regular surface P, we can consider the regular chart $\underline{x}(u, v) = (u, v, 0)$, $(u, v) \in \mathbb{R}^2$. However, this is not the only chart we could use.

(a) Let $U = \mathbb{R}_+ \times (0, 2\pi)$. Show that $\underline{y} : U \to V = P \setminus \{(x, 0, 0) \in \mathbb{R}^3 \mid x \geq 0\}$ defined by $\underline{y}(\rho, \theta) = (\rho \cos\theta, \rho \sin\theta, 0)$, is a system of local coordinates of $V \subset S$.

(b) Compute the coefficients of the first fundamental form of V with respect to the system of local coordinates $\underline{y}$.

(c) Is $\underline{x} \circ \underline{y}^{-1}$ an isometry? Justify your answer.

(d) Compute the Gaussian and principal curvatures with respect to the system of local coordinates $\underline{y}$. How do they compare to the ones computed with respect to $\underline{x}$? Give a geometrical interpretation of the result.

Problem 3. (5 + 13 + 7 = 25 points)
In this problem we are going to show that is possible to think about vector fields as differential operators and relate them to the differential of functions on a surface.
On a surface S, let $X_p \in T_pS$ be a tangent vector and $f : U \subseteq S \to \mathbb{R}$ be a smooth function defined on an open neighbourhood U of p in S. Define the action of the vector on functions as follows:

$$X_p f := \frac{d}{dt}(f \circ \alpha)\Big|_{t=0}$$

for some curve $\alpha : (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = X_p$.

(a) Show that for any smooth function $f : U \to \mathbb{R}$ and any tangent vector $X_p \in T_pS$ we have $df_p(X_p) = X_p f$.

(b) Let X be a vector field on U and denote by X_p the vector $X(p)$. Show that X is smooth if and only if for any function $f : U \to \mathbb{R}$, the function $p \to X_p f$ is smooth.

(c) Let X, Y, Z be smooth vector fields on S. Then $\langle Y, Z\rangle : p \mapsto \langle Y_p, Z_p\rangle_p$ is a smooth function $S \to \mathbb{R}$. Show that for any $p \in S$ we have

$$X_p\langle Y, Z\rangle = \langle (D_X Y)(p), Z_p\rangle_p + \langle Y_p, (D_X Z)(p)\rangle_p$$

Problem 4. (5 + 7 + 5 + 3 = 20 points)
Let S be a surface of revolution, let $\delta : S \to \mathbb{R}$ be the distance of a point $p \in S$ from the axis of rotation, and let θ be the angle between α' and meridians of S.
Clairaut's theorem says that if α is a geodesic, then $\delta \sin\theta$ is constant along α. Viceversa, if $\delta \sin\theta$ is constant along α and no part of α is part of a parallel of S, then α is a geodesic.

We are going to use Clairaut's theorem to find the geodesics of the so-called pseudosphere PS. Consider the parametrisation

$$\underline{x}(u, v) = \left(\frac{1}{v}\cos u, \frac{1}{v}\sin u, \sqrt{1 - v^{-2}} - \operatorname{acosh} v\right), \quad (u, v) \in (0, 2\pi) \times (1, +\infty).$$

(a) Show that the coefficients of the first fundamental form with respect to $\underline{x}$ are $F = 0$, $E = G = 1/v^2$. *Hint: in our domain* $(\operatorname{acosh} x)' = (x^2 - 1)^{-1/2}$.

(b) Use Clairaut's theorem on a unit speed geodesic $\alpha : I \to PS$ with $\alpha(t) = \underline{x}(u(t), v(t))$ to show that geodesics must satisfy the ODE

$$u' = Cv^2$$

for some constant C.

(c) If $C \neq 0$, show that the previous ODE implies that geodesics must satisfy

$$(u - u_0)^2 + v^2 = \frac{1}{C^2}$$

for some constant u_0.

(d) Give a geometric description of these geodesics as images of curves from the uv-plane to the surface.

Solutions

Problem 1

(a) This follows directly from the definition of torsion. When $\kappa \neq 0$, the normal vector is well defined, and $\tau(s) \equiv 0$ implies $\underline{b}'(s) \equiv 0$, or equivalently that the binormal vector $\underline{b}$ is constant.

(b) Point (a) implies that $(\alpha(s) \cdot \underline{b}(s))' = \alpha'(s) \cdot \underline{b}(s) = \underline{t}(s) \cdot \underline{b} \equiv 0$, and therefore there is some constant $C \in \mathbb{R}$ such that $\alpha(s) \cdot \underline{b}(s) \equiv C$: i.e. α is contained on a plane normal to the constant vector $\underline{b}$.

(c) Use the hint and introduce the equation for the center of the osculating circle $c(s)$: $\alpha(s) + \kappa^{-1}\underline{n}(s)$ (see Exercise 1-6.2.b). Being the curvature κ constant and $\tau(s) \equiv 0$, one has that

$$c'(s) = \underline{t}(s) + \kappa^{-1}\underline{n}'(s) = \underline{t}(s) + \kappa^{-1}\left(-\kappa\underline{t}(s) - \tau\underline{b}(s)\right) = 0,$$

and therefore $c(s) \equiv c$ is a constant vector. The statement follows by observing that $\|\alpha - c\|^2 = \| - \kappa^{-1}\underline{n}(s)\|^2 = \kappa^{-2}$ is the equation of a sphere of radius κ^{-1} centred at c.

(d) A direct consequence of (b) and (c) is that α is a parametrisation for an arc of the circle C obtained intersecting the sphere S and the plane P. A circle is the osculating circle of itself, so its radius is exactly the inverse of its curvature. This means that the radius of C is κ^{-1}, and thus P has to intersect S through its centre.

Wrapping it up: C is the circle of radius κ^{-1}, centred at c and contained in the plane $P = \{v \in \mathbb{R}^3 \mid (v - c) \cdot \underline{b} = 0\}$, i.e. the plane containing c and orthogonal to $\underline{b}$.

Problem 2

(a) We need to check the three properties in the definition of regular surface (Definition 1 of Chapter 2-2, on page 54). One way of doing it is the following.

1. Differentiability: the chart is made of product of differentiable function.
3. The differential of $\underline{y}$ is easily computed as

$$d\underline{y}_p = \begin{pmatrix} | & | \\ \underline{y}_\rho & \underline{y}_\theta \\ | & | \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\rho\sin(\theta) \\ \sin(\theta) & \rho\cos(\theta) \\ 0 & 0 \end{pmatrix},$$

which is clearly of rank 2 for $\rho > 0$, and thus injective on U.

(a) Homeomorphism: the map $(\rho,\theta) \mapsto (\rho\cos(\theta), \rho\sin(\theta), 0)$ is one-to-one on U. The property is satisfied if $\underline{y}^{-1}$ is continuous: this follows as a consequence of Chapter 1-2, Proposition 4 (on page 66).

(b) We can use $\underline{y}_\rho$, $\underline{y}_\theta$ computed at the point above to immediately get

$$E = \left\langle \underline{y}_\rho, \underline{y}_\rho \right\rangle = \cos^2(\theta) + \sin^2(\theta) = 1,$$
$$F = \left\langle \underline{y}_\rho, \underline{y}_\theta \right\rangle = -\rho\sin(\theta)\cos(\theta) + \rho\cos(\theta)\sin(\theta) = 0,$$
$$G = \left\langle \underline{y}_\theta, \underline{y}_\theta \right\rangle = \rho^2\sin^2(\theta) + \rho^2\cos^2(\theta) = \rho^2.$$

(c) The coefficients $\tilde{E} = 1$, $\tilde{F} = 0$, $\tilde{G} = 1$ of the first fundamental form for $\underline{x}$ can be immediately computed from the definition, otherwise one can refer to Chapter 2-5, Example 1 (on pages 95-96).

We know that $\underline{x} \circ \underline{y}^{-1}$ is an isometry iff $\phi := \underline{y} \circ \underline{x}^{-1}$ is an isometry. Assume that ϕ is an isometry. Given that $\underline{y} = \phi \circ \underline{x}$, Exercise 4-2.2, on page 231, would imply in particular that $1 = \tilde{G} = G = \rho^2$, providing a contradiction.

It follows that ϕ (and thus $\underline{x} \circ \underline{y}^{-1}$) is not an isometry.

(d) To compute the Gaussian curvature K and the principal curvatures k_1 and k_2 one can use the equations (4) and (6) on page 158. Otherwise one can directly reason about the differential of the Gauss map.

In any case it is immediate to compute

$$N(p) = \frac{\underline{y}_\rho \wedge \underline{y}_\theta}{\left\| \underline{y}_\rho \wedge \underline{y}_\theta \right\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\underline{x}_\rho \wedge \underline{x}_\theta}{\left\| \underline{x}_\rho \wedge \underline{x}_\theta \right\|}$$

and therefore $dN_p = 0$. In particular, this means that in both cases

$$K(p) = \det(dN_p) = k_1 k_2 = 0 \quad \text{and} \quad H(p) = -\operatorname{tr}(dN_p)/2 = \frac{k_1 + k_2}{2} = 0,$$

and therefore $k_1 = k_2 = 0$.

Problem 3

(a) Let $\alpha : I \to S$ be a smooth curve with $\alpha(0) = p$ and $\alpha'(0) = X_p$. As f is smooth, we can apply the chain rule to get

$$X_p f = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = df_{\alpha(0)} \circ \alpha'(0) = df_p(X_p).$$

(b) Assume X is a smooth vector field on U. Let $\underline{x} : W \subseteq \mathbb{R}^2 \to V \subseteq U$ a regular parametrisation around $p \in U$. Then for all $q \in V$ there are two smooth functions $a, b : V \to \mathbb{R}$ such that in coordinates

$$X_q = a(u,v)\underline{x}_u + b(u,v)\underline{x}_v.$$

If α is a curve like the one defined in the previous point, and $f : U \to \mathbb{R}$ is a smooth map, by (a) one has that for all $q \in V$

$$X_q(f \circ \underline{x}) = df_q(X_q) = df_q\left(a(u,v)\underline{x}_u + b(u,v)\underline{x}_v\right)$$

which is smooth as it is the composition of smooth functions.

On the other hand, let X be a vector field, and assume that the mapping $q \mapsto X_q$ is smooth. Then, again, in coordinates

$$X_q = a(u,v)\underline{x}_u + b(u,v)\underline{x}_v.$$

for some regular parametrisation $\underline{x} : W \subseteq \mathbb{R}^2 \to V \subseteq U$ around $p \in U$. We want to show that $a, b : V \to \mathbb{R}$ are smooth.

Let α be a curve defined as in (a). Then, in coordinates, for all $q \in V$ we have

$$X_q(f \circ \underline{x}) = df_q(X_q) = df_q\left(a(u,v)\underline{x}_u + b(u,v)\underline{x}_v\right) = a(u,v)(f \circ \underline{x})_u + b(u,v)(f \circ \underline{x})_v.$$

Since $q \mapsto X_q$ is smooth, also the coefficients a, b have to be smooth.

(c) Let once more $\alpha : I \to S$ be a smooth curve with $\alpha(0) = p$ and $\alpha'(0) = X_p$, as in (a). Then, Exercise 4-4.4 on page 263 implies

$$\begin{aligned} X_p\langle Y, Z\rangle &= \frac{d}{dt}\left(\langle Y, Z\rangle \circ \alpha\right)\Big|_{t=0} \\ &= \frac{d}{dt}\langle Y(t), Z(t)\rangle\Big|_{t=0} \\ &= \left\langle \frac{DY}{dt}(0), Z_p \right\rangle + \left\langle Y_p, \frac{DZ}{dt}(0)\right\rangle \\ &= \langle (D_X Y)(p), Z_p\rangle_p + \langle Y_p, (D_X Z)(p)\rangle_p. \end{aligned}$$

Problem 4

(a) The differential of $\underline{x}$ is

$$d\underline{x}_p = \begin{pmatrix} | & | \\ \underline{x}_u & \underline{x}_v \\ | & | \end{pmatrix} = \begin{pmatrix} -\frac{1}{v}\sin(u) & -\frac{1}{v^2}\cos(u) \\ \frac{1}{v}\cos(u) & -\frac{1}{v^2}\sin(u) \\ 0 & -\frac{\sqrt{v^2-1}}{v^2} \end{pmatrix},$$

from which it immediately follows that

$$\begin{aligned} E &= \langle \underline{x}_u, \underline{x}_u\rangle = \frac{1}{v^2}\sin^2(u) + \frac{1}{v^2}\cos^2(u) = \frac{1}{v^2}, \\ F &= \langle \underline{x}_u, \underline{x}_v\rangle = \frac{1}{v^3}\sin(u)\cos(u) - \frac{1}{v^3}\cos(u)\sin(u) = 0, \\ G &= \langle \underline{x}_v, \underline{x}_v\rangle = \frac{1}{v^4}\cos^2(u) + \frac{1}{v^4}\sin^2(u) + \frac{v^2-1}{v^4} = \frac{1}{v^2}. \end{aligned}$$

(b) Using (a), we have that

$$1 = |\alpha'(t)|^2 = \mathrm{I}_p(\alpha'(t)) = \frac{(u')^2(t) + (v')^2(t)}{v^2(t)}.$$

Thus, the components of α satisfy the ODE

$$(u')^2(t) + (v')^2(t) = v^2(t). \tag{1}$$

At the same time, since the distance of $\alpha(t)$ from the z-axis is $1/v(t)$, Clairaut's theorem applied to α gives

$$C = \frac{1}{v(t)}\sin(\theta(t)) \tag{2}$$

for some constant C. Since $\underline{x}_v$ is the vector parallel to the meridians, we also have

$$\cos(\theta) = \frac{\langle \alpha', \underline{x}_v\rangle}{\sqrt{G}} = \frac{v'}{v},$$

from which one gets

$$\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \sqrt{\frac{v^2 - (v')^2}{v^2}} = \frac{u'}{v},$$

where we used (1) for the last step. Replacing $\sin(\theta)$ in (2) with the value just obtained, one gets

$$C = \frac{u'}{v^2}$$

concluding the proof.

(c) Substituting the equation in (b) inside (1) we get $(Cv^2)^2 + (v')^2 = v^2$, i.e.

$$v' = \pm v\sqrt{1 - C^2 v^2}.$$

Thus, along the geodesic, we can use the equation in (b) and (1) to get

$$\frac{u'}{v'} = \pm\frac{Cv}{\sqrt{1 - C^2 v^2}},$$

which remarkably is a separable ODE. Multiplying by v' and integrating both sides with respect to t we get

$$u - u_0 = \mp\frac{1}{C}\sqrt{1 - C^2 v^2}$$

for some constant u_0. To conclude the proof, is enough to square both sides of the equation and reorder the terms.

(d) First of all observe that we can immediately see from the equation in (b) that for $C = 0$ we get the meridians $u(t) \equiv u_0$. The corresponding geodesics are the images under $\underline{x}$ of such vertical straight lines in the vw-plane.

The equation in point (c) on the other hands tells us that, for $C \neq 0$ the geodesics are images of circles centred at $(u_0, 0)$, lying in the region $v > 1$. Note in particular that such circles have centre on the v-axis, and so intersect it perpendicularly.